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ADP010895 thru ADP010929

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SYMMETRIC FRACTALS GENERATED BY CELLULAR AUTOMATA

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We demonstrate how to construct fractals which are generated by a combination of a cellular automaton and a substitution. Moreover, if the substitution and the cellular automaton exhibit certain symmetry features, the fractal will inherit these symmetries.

Keywords: fractals, cellular automata, substitutions, symmetry, invariants

1 Introduction

It is a well known fact that certain cellular automata can generate fractals^{7,8,9,14,15}. These fractals can be described in terms of hierarchical iterated function systems¹², graph directed constructions¹¹ or as a mixed self-similar set².

In this note we consider cellular automata which have certain symmetries. After the introduction of some basic concepts and formalism about substitutions, fractals and cellular automata in Sections 2 and 3, we shall show in Section 4 how to construct fractals generated by a cellular automaton such that the fractal inherits the symmetries of the cellular automaton.

2 Substitutions and fractals

In this section we introduce the concept of a substitution which is needed for our applications. We discuss how substitutions generate a compact subset which exhibits a hierarchical self-similar structure, and thus is usually a fractal set.

We will consider sequences $(f_j)_{j \in \mathbb{Z}}$, where f_j belongs to a finite commutative ring \mathcal{R} with 0 and 1. It will be useful to represent this sequence as a formal Laurent series with coefficients in \mathcal{R} , as \underline{f} or as

$$\underline{f} = f(X) = \sum_{j \in \mathbb{Z}} f_j X^j.$$

The set of all such Laurent series will be denoted by $\mathcal{R}(X)$. The support $\text{supp}(\underline{f})$ of $\underline{f} \in \mathcal{R}(X)$ is the set $\{j \mid f_j \neq 0\}$. \underline{f} is called a Laurent polynomial if $\text{supp}(\underline{f})$ is finite. The set of all Laurent polynomials is denoted by $\mathcal{R}_c(X)$. If $\text{supp}(\underline{f}) \subset \mathbb{N}$ and is finite, then \underline{f} is called a polynomial. The set of polynomials with coefficients in \mathcal{R} is denoted by $\mathcal{R}[X]$.

The set of all maps $a : \mathcal{R} \rightarrow \mathcal{R}$ equipped with the addition and composition is denoted by $\text{Abb}(\mathcal{R})$. With 0 we denote the map $r \mapsto 0$ and with 1 we denote the identity map $r \mapsto r$. The subset of all maps $a : \mathcal{R} \rightarrow \mathcal{R}$ with $a(0) = 0$ is denoted as $\text{Abb}_0(\mathcal{R})$. If $a \in \text{Abb}(\mathcal{R})$, then a induces a map, also denoted by a , from $\mathcal{R}(X)$ to $\mathcal{R}(X)$ which is defined as $a(\underline{f})(X) = \sum_{j \in \mathbb{Z}} a(f_j) X^j$.

For $k \in \mathbb{N}$ the set $\{0, \dots, k-1\}$ is denoted by $[k]$.

A k -substitution transforms a sequence $(f_j)_{j \in \mathbb{Z}}$ into a new sequence by replacing each element f_j by a string of k elements $\xi_0(f_j)\xi_1(f_j)\dots\xi_{k-1}(f_j)$, where $\xi_j \in \text{Abb}(\mathcal{R})$. Here follows the formal

Definition 2.1 A k -substitution ξ is a k -tuple $\xi = (\xi_l)_{l \in [k]} \in (\text{Abb}(\mathcal{R}))^k$ which defines a map ξ on $\mathcal{R}(X)$ given by

$$\xi(\underline{f}) = \sum_{l=0}^{k-1} X^l \xi_l(\underline{f})(X^k),$$

where $\xi_j(\underline{f})(X) = \sum_{j \in \mathbb{Z}} \xi_l(f_j)X^j$. If $\xi \in (\text{Abb}_0(S))^k$, i.e., when 0 is replaced by a string of k 0's, then ξ is called a regular k -substitution.

Remark 1. A k -substitution $\xi = (\xi_l)_{l \in [k]}$ is also defined by a polynomial $P_\xi(X) = \sum \xi_j X^j \in \text{Abb}(\mathcal{R})[X]$ of degree less than k and we write $\xi(\underline{f}) = P_\xi(X) \circ f(X^k)$, where the product is the product induced by the composition of maps and the coefficients f_j of $f(X)$ are considered as constant maps $r \mapsto f_j$. The polynomial P_ξ is called the *substitution polynomial*.

2. If $Q(X) \in \mathcal{R}[X]$ is a polynomial of degree less than or equal to $k-1$, then $\xi_Q(\underline{f}) = Q(X)f(X^k)$ defines a k -substitution with $\xi_j(r) = q_j r$.

3. If ξ_1 and ξ_2 are k -substitutions, respectively, and if $P_{\xi_1}(X)$ and $P_{\xi_2}(X)$ denote the respective substitution polynomials, then $\xi = \xi_1 \circ \xi_2$ is a k^2 substitution with substitution polynomial $P_\xi(X) = P_{\xi_1}(X) \circ P_{\xi_2}(X^k)$.

We also need to consider two-dimensional sequences $(g_{i,j})_{i,j \in \mathbb{Z}}$ with $g_{i,j} \in \mathcal{R}$. The corresponding formal Laurent series representation is

$$\underline{g} = f(X, Y) = \sum_{i,j \in \mathbb{Z}} g_{i,j} X^i Y^j,$$

$\mathcal{R}(X, Y)$ is the set of all such two-dimensional Laurent series (sequences). Like for the one-dimensional case, we say that $g(X, Y)$ is a Laurent polynomial if $g_{i,j} = 0$ almost everywhere. Like for the one-dimensional case, two-dimensional sequences can be transformed by a two-dimensional $(k \times k)$ -substitution, which replaces each symbol $g_{i,j}$ by a $(k \times k)$ -array of symbols in \mathcal{R} . Formally:

Definition 2.2 Let $\Xi = (\xi_{l,m})_{l,m \in [k]} \in (\text{Abb}(\mathcal{R}))^{k \times k}$. The map $\Xi : \mathcal{R}(X, Y) \rightarrow \mathcal{R}(X, Y)$ defined as

$$\Xi(\underline{g}) = \sum_{l,m=0}^{k-1} X^l Y^m \xi_{l,m}(\underline{g})(X^k, Y^k)$$

is called a $(k \times k)$ -substitution. It is called regular if $\Xi \in (\text{Abb}_0(\mathcal{R}))^{k \times k}$.

Let us agree here, that in two-dimensional sequences and arrays, and in the corresponding graphical representations, the first index, which is possibly associated to the symbol X in formal Laurent series, increases along a horizontal axis which is oriented from the left to the right; while the second index, possibly associated to the symbol Y , increases along a vertical axis which is oriented from top to bottom.

Remark 1. If Ξ is a regular $(k \times k)$ -substitution, then $\Xi(\mathcal{R}_c(X)) \subseteq \mathcal{R}_c(X)$. In other words, the set of Laurent polynomials is invariant under regular k - or $(k \times k)$ -substitutions.

2. If Ξ_1 and Ξ_2 are $(k \times k)$ -substitutions, respectively, and P_{Ξ_1}, P_{Ξ_2} the respective substitution polynomials, then $\Xi_1 \circ \Xi_2$ is a $(k^2 \times k^2)$ -substitution with substitution polynomial $P_{\Xi_1 \circ \Xi_2}(X, Y) = P_{\Xi_1}(X, Y) \circ P_{\Xi_2}(X^k, Y^k)$.

In order to draw the connection between fractals and substitutions we define a graphical representation of elements in $\mathcal{R}_c(X)$ or in $\mathcal{R}_c(X, Y)$. We only need to introduce the graphical representation for elements in $\mathcal{R}_c(X, Y)$, since any Laurent series in $\mathcal{R}(X)$ has a natural counterpart in $\mathcal{R}(X, Y)$.

Let $(\mathcal{H}(\mathbb{R}^2), d_H)$ denote the set of non-empty compact subsets of \mathbb{R}^2 , where d_H is the Hausdorff distance induced by the Euclidian metric (or any other equivalent metric) on \mathbb{R}^2 . Since \mathbb{R}^2 is a complete metric space, so is $(\mathcal{H}, d_H)^{5,6}$.

Definition 2.3 Let $I(i, j) = [i, i + 1] \times [j, j + 1]$. Then the map $\mathbf{G} : \mathcal{R}_c(X, Y) \rightarrow \mathcal{H}(\mathbb{R}^2) \cup \{\emptyset\}$ defined as

$$\mathbf{G}(\underline{g}) = \{I(i, j) \mid g_{i,j} \neq 0\}$$

is called graphical representation.

$I(i, j)$ in the definition above can be considered as a “highlighted pixel” at location (i, j) on a graphical display which coincides with the \mathbb{R}^2 -plane (as agreed: y-axis positively oriented downwards).

Remark 1. $\mathbf{G}(\underline{g}) = \emptyset$ if and only if $\underline{g} = 0$.

2. Any map $\gamma : \mathcal{R} \setminus \{0\} \rightarrow \mathcal{H}(\mathbb{R}^2)$ induces another graphical representation \mathbf{G}_γ by defining $\mathbf{G}_\gamma(\underline{g}) = \cup \{(i, j) + \gamma(g_{i,j}) \mid g_{i,j} \neq 0\}$. This corresponds to “non-square pixels”, where the pixel-shape may depend on $g_{i,j}$.

3. The concept of graphical representation is quite general: when \mathcal{S} is a ring with 0 and 1, then so is $\mathcal{R} = \mathcal{S}^D$, with $0_{\mathcal{R}} = (0, 0, 0, \dots, 0)$ and $1_{\mathcal{R}} = (1, 1, 1, \dots, 1)$ (both $0_{\mathcal{R}}$ and $1_{\mathcal{R}}$ have D components). So it is possible to speak about the graphical representation of a two-dimensional sequence with values in $\mathcal{R} = \mathcal{S}^D$, where $g_{i,j}$ is just an element of \mathcal{S}^D .

The connection between fractals and substitutions is provided by the following theorem.

Theorem 2.4 ⁸ Let $\Xi : \mathcal{R}(X, Y) \rightarrow \mathcal{R}(X, Y)$ be a regular $(k \times k)$ -substitution such that for all $r \neq 0$, there are i, j such that $\xi_{i,j}(r) \neq 0$. If $\underline{g} \in \mathcal{R}_c(X, Y)$ and $\underline{g} \neq 0$, then the sequence

$$\left(\frac{1}{k^n} \mathbf{G}(\Xi^n(\underline{g})) \right)_{n \in \mathbb{N}}$$

is a Cauchy sequence in (\mathcal{H}, d_H) .

The proof relies on two facts. Firstly, Ξ being regular maps Laurent polynomials on Laurent polynomials. Thus the above sequence is indeed in $\mathcal{H}(\mathbb{R}^2)$. Secondly, again due to the regularity of Ξ one has $\frac{1}{k} \mathbf{G}(\Xi(\underline{g})) \subset \mathbf{G}(\underline{g})$ for all Laurent polynomials different from 0. These two observations conclude the proof.

Remark 1. The limit of the above Cauchy sequence is denoted by $\mathcal{A}(\underline{g})$.

2. If \mathbf{G}_γ is another graphical representation, then the above theorem remains true. Moreover, the limit set does not depend on the graphical representation.

3. For $\underline{g} \in \mathcal{R}_c(X, Y)$, $\underline{g} \neq 0$, one has

$$\mathcal{A}(\underline{g}) = \bigcup_{\{(i,j) \mid g_{i,j} \neq 0\}} \mathcal{A}(g_{i,j}) + (i, j),$$

where $\mathcal{A}(g_{i,j}) + (i, j)$ is the translated limit set of the graphical representation of $g(X, Y) = g_{i,j}$.

Example 1. Let $\mathcal{R} = \mathbb{Z}_2$ and let the 3-substitution be given by the substitution polynomial $P_\xi(X) = \text{id} + \text{id } X^2$, where id denotes the 1 in $\text{Abb}(\mathcal{R})$. The limit set $\mathcal{A}(1)$, i.e. the limit of the properly rescaled graphical representation of the sequence $\cdots 0001000 \cdots$ under the substitution $0 \mapsto 000$, $1 \mapsto 101$ is the triadic Cantor set.

2. Let $\mathcal{R} = \mathbb{Z}_2$ and define the (2×2) -substitution by the substitution polynomial $P_\Xi(X, Y) = \text{id} + \text{id } Y + \text{id } XY$. The limit set $\mathcal{A}(1)$, i.e. the limit of the properly rescaled graphical representation of the sequence $g(X, Y) = 1$ (a single value 1 at $(i, j) = (0, 0)$) under the substitution $0 \mapsto \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$, $1 \mapsto \begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}$, is the Sierpinski triangle.

3 Cellular automata

Substitutions transform a sequence into another sequence. So do other mechanisms known as cellular automata. We review briefly some essentials which we need further.

Let \mathcal{S} denote a commutative ring with 0 and 1. According to Hedlund¹⁰, a cellular automaton with states in the ring \mathcal{S} (actually it is only necessary that \mathcal{S} is a finite set) is a continuous map $A : \mathcal{S}(X) \rightarrow \mathcal{S}(X)$ which commutes with the shift map. This implies that every cellular automaton A is given by a local rule $\phi : \mathcal{S}^{N_1+N_2} \rightarrow \mathcal{S}$, where $N_1, N_2 \in \mathbb{N}$ and A is defined as

$$A(\underline{f}) = \sum_{j \in \mathbb{Z}} \phi(f_{j-N_1}, f_{j-N_1+1}, \dots, f_{j+N_2-1}, f_{j+N_2}) X^j.$$

This means that the j -th element of $A(\underline{f})$ is obtained from \underline{f} as a function ϕ of the f -elements in a fixed finite neighbourhood of j . For a cellular automaton $A : \mathcal{S}(\bar{X}) \rightarrow \mathcal{S}(X)$ and an initial configuration $\underline{f} \in \mathcal{S}(X)$ we define the orbit $O_A(\underline{f})$ of \underline{f} (w.r.t. A) as the two-dimensional sequence defined on $\mathbb{Z} \times \mathbb{N}$, where row $t \in \mathbb{N}$ displays the t -th iterate of A on \underline{f} . Formally, and in Laurent series notation:

$$O_A(\underline{f})(X, Y) = \sum_{t=0}^{\infty} A^t(\underline{f}(X)) Y^t.$$

In this note we are dealing with cellular automata $A = A_\phi$ with local rule $\phi : \mathcal{S}^D \rightarrow \mathcal{S}$ such that

$$A_\phi(\underline{f})(X) = \sum_{j \in \mathbb{Z}} \phi(f_{j-D+1}, \dots, f_j) X^j.$$

Of particular interest are linear cellular automata¹, i.e., automata with a \mathcal{S} -linear local rule $\phi : \mathcal{S}^D \rightarrow \mathcal{S}$. In this case the cellular automaton is also described by the multiplication with a "local-rule"-polynomial $R(X)$, i.e., $A_\phi(\underline{f}) = R(X)f(X)$.

We are particularly interested in cellular automata with the additional property of being k -Fermat⁹. They provide a framework to explain the existence of fractals which are generated by cellular automata.

For $k \in \mathbb{N} \setminus \{0\}$ we define the k -th power $\pi_k : \mathcal{S}(X) \rightarrow \mathcal{S}(X)$ as $\pi_k(\underline{f})(X) = \sum f_j X^{jk} = f(X^k)$, i.e., π_k "inflates" \underline{f} by inserting $(k-1)$ 0's between the elements of the sequence \underline{f} (we could call this " k -inflation").

We now recall the concept of a cellular automaton having the k -Fermat property. This implies that row k of the orbit of a " k -inflated" initial sequence equals the " k -inflation" of row 1 of the orbit of the "non-inflated" initial sequence, no matter what the initial sequence might be. Formally:

Definition 3.1 ^{1,9} A cellular automaton $A : \mathcal{S}(X) \rightarrow \mathcal{S}(X)$ is called k -Fermat if $(A^k \circ \pi_k)(\underline{f}) = (\pi_k \circ A)(\underline{f})$ holds for all $\underline{f} \in \mathcal{S}(X)$.

Remark 1. If A is a k -Fermat cellular automaton, then the sequence 0 is a fixed point of A . Therefore $A(\mathcal{S}_c(X)) \subseteq \mathcal{S}_c(X)$ for a k -Fermat cellular automaton.

2. If A is a linear cellular automaton defined by the Laurent polynomial $R(X)$, then A is k -Fermat if and only if $R(X)^k = R(X^k)$.

Example 1. Let $\mathcal{S} = \mathbb{F}_{p^\alpha}$, the field with p^α elements and of characteristic p (p being a prime number). Then any linear cellular automaton is p^α -Fermat.

2. Let $\mathcal{S} = \mathbb{Z}_{p^\alpha}$, p a prime number. A linear cellular automaton is p -Fermat if its local rule is a Laurent polynomial $R(X)$ of the form $R(X) = Q(X)p^{\alpha-1}$, where $Q(X)$ is a Laurent polynomial.

3. Let $\mathcal{S} = \mathbb{Z}_6$, then $R(X) = 3 + 4X$ defines a 2-Fermat linear cellular automaton and $R(X) = 2 + 4X$ defines a 3-Fermat linear cellular automaton.

4. Let $\mathcal{S} = \mathbb{Z}_5$. Then the local rule $\phi : \mathbb{Z}_5^2 \rightarrow \mathbb{Z}_5$ defined by the table with entries $\phi(x, y)$

$x \backslash y$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	4	2	3
2	2	3	0	4	1
3	3	4	1	0	2
4	4	2	3	1	0

induces a (nonlinear) 2-Fermat cellular automaton. In fact, it can be shown that there are 5^{12} different local rules $\phi : \mathbb{Z}_5^2 \rightarrow \mathbb{Z}_5$ which define a 2-Fermat cellular automaton on $\mathbb{Z}_5(X)$. Which demonstrates that not all k -fermat cellular automata have to be linear cellular automata.

We now present a generic example which illustrates a general property of k -Fermat cellular automata. Consider the cellular automaton with $\mathcal{S} = \mathbb{Z}_3$ and local rule $R(X) = 1 + 2X + X^2 \in \mathbb{Z}_3[X]$. It is 3-Fermat. Part of the orbit $O_A(1)$ is shown below:

```

row 00    ...00010000000000000000000000000000...
row 01    ...00012100000000000000000000000000...
row 02    ...00011011000000000000000000000000...
row 03    ...00010020010000000000000000000000...
row 04    ...00012121212100000000000000000000...
row 05    ...00011000000011000000000000000000...
row 06    ...00010010000010010000000000000000...
row 07    ...00012112100012112100000000000000...
row 08    ...00011022011011022011000000000000...
row 09    ...00010000000020000000010000000000...
row 10    ...00012100000024200000012100000000...
row 11    ...00011011000022022000011011000000...
row 12    ...00010020010020010020010020010000...
row 13    ...0001212121212121212121212121000...

```

Row $3n + l$, $l \in \{0, 1, 2\}$ in this orbit, i.e. the sequence $A^{3n+l}(1)$, can be obtained from row $3n$ as $A^l(A^{3n}(1))$. By the 3-Fermat-property, row $3n$ is a “3-inflated” version of row n (say $\cdots zabcd\cdots$), as given by the top sequence in the following scheme:

row $3n$	$\cdots 0\ 0\ z\ 0\ 0$	$a\ 0\ 0$	$\bigg $	$b\ 0\ 0$	$\bigg $	$c\ 0\ 0\ d\ 0\ 0\ \cdots$
row $3n + 1$	$\cdots 2z\ z$	$a\ 2a\ a$	$\bigg $	$b\ 2b\ b$	$\bigg $	$c\ 2c\ c\ d\ 2d\ d\ \cdots$
row $3n + 2$	\cdots	$a + z\ a + z\ 0$	$\bigg $	$a + b\ a + b\ 0$	$\bigg $	$b + c\ b + c\ 0\ c + d\ c + d\ 0\ \cdots$

(1)

Also represented in this scheme are rows $3n + 1$ and $3n + 2$, obtained by application of the rule $R(X)$. Observe from this scheme that knowledge of two successive elements in row n , say (a, b) , is sufficient to determine all highlighted values. Now define the substitution Ψ which replaces the pair (a, b) by a (3×3) -array of pairs, as follows:

$$(a, b) \mapsto \begin{bmatrix} (0, b) & (b, 0) & (0, 0) \\ (a, b) & (b, 2b) & (2b, b) \\ (0, a + b) & (a + b, a + b) & (a + b, 0) \end{bmatrix}.$$

The last components of the pairs in this array are given by the elements of the 3×3 -array indicated in the above three sequences (1); the first components are the elements that immediately precede them. It is now clear that, if we replace each element (say b) in row n by the corresponding last components in row l ($l = 0, 1, 2$) of the substitution array, we produce row $3n + l$. In that way, the orbit above corresponds to the last elements of the pairs in the arrays obtained by iterating the substitution starting from the sequence of pairs $\cdots (0, 0) (0, 1) (1, 0) (0, 0) \cdots$ (i.e., row 0 rewritten as pairs in which the last components correspond to the elements of row 0, and the first components to the corresponding preceding elements). We show the first step of the iteration (elements of the actual orbit are underlined):

$$\cdots (0, \underline{0})(0, \underline{1})(1, \underline{0})(0, \underline{0}) \cdots$$

$$\downarrow$$

$$\cdots \left| \begin{array}{c} (0, \underline{0})(0, \underline{0})(0, \underline{0}) \\ (0, \underline{0})(0, \underline{0})(0, \underline{0}) \\ (0, \underline{0})(0, \underline{0})(0, \underline{0}) \end{array} \right| \left| \begin{array}{c} (0, \underline{1})(1, \underline{0})(0, \underline{0}) \\ (0, \underline{1})(1, \underline{2})(2, \underline{1}) \\ (0, \underline{1})(1, \underline{1})(1, \underline{0}) \end{array} \right| \left| \begin{array}{c} (0, \underline{0})(0, \underline{0})(0, \underline{0}) \\ (1, \underline{0})(0, \underline{0})(0, \underline{0}) \\ (0, \underline{1})(1, \underline{1})(1, \underline{0}) \end{array} \right| \left| \begin{array}{c} (0, \underline{0})(0, \underline{0})(0, \underline{0}) \\ (0, \underline{0})(0, \underline{0})(0, \underline{0}) \\ (0, \underline{0})(0, \underline{0})(0, \underline{0}) \end{array} \right| \cdots$$

In the next step, each pair is again subjected to the substitution Ψ . In general, the pairs will be D -tuples, and the substitution will replace D -tuples by $(k \times k)$ -arrays of D -tuples (for a k -Fermat cellular automaton). We now express this property as a formal theorem.

Let D be a natural number, $D \geq 1$. We define an embedding $\iota = \iota_D$ of $S(X)$ into $S^D(X)$ as follows ($S^D(X)$ can be considered as the set of sequences with D -tuples as values). For $f(X) = \sum_{j \in \mathbb{Z}} f_j X^j \in S(X)$ we define $\iota(f) \in S^D(X)$ as

$$\iota(\underline{f})(X) = \sum_{j \in \mathbb{Z}} (f_{j-D+1}, \dots, f_j) X^j.$$

We also define a projection $\mathbf{p} = \mathbf{p}_D : S^D(X) \rightarrow S(X)$ such that $\mathbf{p} \circ \iota(f) = f$ by setting $\mathbf{p} \left(\sum_{j \in \mathbb{Z}} (s_{j,-D+1}, s_{j,-D+2}, \dots, s_{j,0}) X^j \right) = \sum_{j \in \mathbb{Z}} s_{j,0} X^j$.

With these notions the work of several authors^{7,8,9,13,14} on fractals generated by cellular automata can be summarized as

Theorem 3.2 *If $A : S(X) \rightarrow S(X)$ is a k -Fermat cellular automaton, then there exists a $D \in \mathbb{N}$ and a regular $(k \times k)$ -substitution $\Psi = (\psi_{i,j})_{i,j \in [k]} : S^D(X, Y) \rightarrow S^D(X, Y)$ such that for all $n \in \mathbb{N}$ and $l \in [k]$ we have*

$$A^{nk+l}(1) = \mathbf{p}(\sigma_l(\iota(A^n(1)))) ,$$

where σ_l is the k -substitution $(\psi_{i,l})_{i \in [k]}$.

The interpretation in terms of fractals generated by cellular automata is clear. If A is a k -Fermat cellular automaton, then the initial configuration 1 generates an orbit. If the orbit is visualized by a graphical representation, one observes a certain pattern. According to Theorem 3.2, the pattern can be described by a $(k \times k)$ -substitution $\Psi = (\psi_{i,m})$ and by Theorem 2.4 the pattern is represented by a compact subset. Moreover, for linear k -Fermat cellular automata it is known that the limit set does not depend on the initial configuration $\underline{f} \in S_c(X)$.

4 Cellular automata, symmetries and invariants

It is worthwhile to study cellular automata with a local rule $\phi : S^2 \rightarrow S$ which exhibits certain symmetries^{3,4}.

Definition 4.1 *The map $\phi : S^2 \rightarrow S$ is called rotationally symmetric if $\phi(\phi(x, y), x) = y$ holds for all $x, y \in S$. If ϕ is rotationally symmetric and commutative, i.e., $\phi(x, y) = \phi(y, x)$, then ϕ is called totally symmetric.*

The following geometric interpretation clarifies the definition. We represent the relation $z = \phi(x, y)$ as a triangular array ${}^x z^y$, and call it an elementary ϕ -configuration. In geometric terms the rotational symmetry of ϕ can be phrased as: Any rotation (by $\frac{2\pi}{3}$) of an elementary ϕ -configuration, changing ${}^x z^y$ into ${}^z y^x$ or ${}^y x^z$, is an elementary ϕ -configuration.

The commutativity of ϕ implies the invariance of elementary ϕ -configurations under vertical reflection. Combined with the rotational symmetry of ϕ we can state that ϕ is totally symmetric if and only if the set of elementary ϕ -configurations is invariant under the symmetry group of the equilateral triangle.

Example 1. The local rule $\phi : \mathbb{Z}_2^2 \rightarrow \mathbb{Z}_2$ defined as $\phi(x, y) = x + y$ is totally symmetric.

2. Let $\mathbb{F}_4 = \{0, 1, \zeta, 1 + \zeta\}$ denote the field with 4 elements. The local rule $\phi : \mathbb{F}_4^2 \rightarrow \mathbb{F}_4$ defined as $\phi(x, y) = \zeta x + (1 + \zeta)y$ is rotationally symmetric but not totally symmetric.

3. There exist rotationally symmetric local rules which are not related to linear rules.⁴

Definition 4.2 A ϕ -configuration of size N is a top down equilateral triangular array with N elements in the top row and with values in S such that any subtriangle ${}^x y^z$ is an elementary ϕ -triangle.

A ϕ -configuration of size N is rotationally symmetric if the rotated ϕ -configuration is equal to the ϕ -configuration.

A ϕ -configuration of size N is totally symmetric if it remains unchanged under the symmetry group of the equilateral triangle.

Figure 1 shows examples of ϕ -configurations.

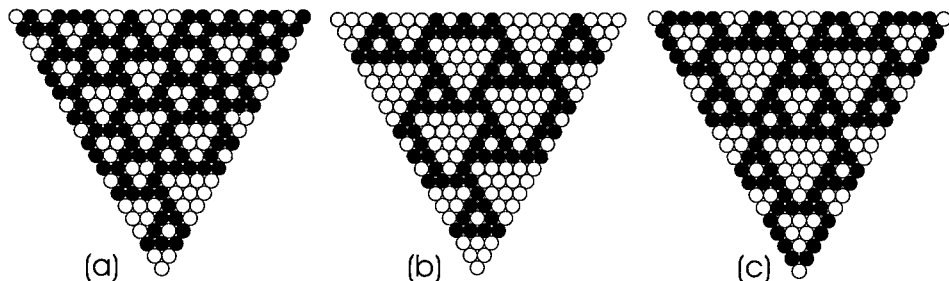


Figure 1. Examples of ϕ -configurations of size 21 for $\phi(x, y) = x + y$ defined in \mathbb{Z}_2 .

(a): a nonsymmetric one, (b): a rotationally symmetric one, (c): a totally symmetric one. (A black cell represents the value 1, a white cell the value 0).

Theorem 4.3⁴ If $\phi : S^2 \rightarrow S$ is rotationally symmetric and of the form $\phi(x, y) = rx + sy$, then there exist for any $N \in \mathbb{N}$ a rotationally symmetric ϕ -configuration of size N .

If $\phi : S^2 \rightarrow S$ is totally symmetric and of the form $\phi(x, y) = rx + sy$, then there exists for any $n \in \mathbb{N}$ a totally symmetric ϕ -configuration of size N .

As a next step we introduce invariants of cellular automata. The motivating example for invariants is given by the k -Fermat cellular automata. Note that the map $\pi_k : \mathcal{S}(X) \rightarrow \mathcal{S}(X)$ is a special kind of k -substitution. In fact, the substitution polynomial associated with π_k is $P_{\pi_k}(X) = \text{id}$ and $A^k \circ \pi_k = \pi_k \circ A$. It is therefore natural to define

Definition 4.4 Let $A : \mathcal{S} \rightarrow \mathcal{S}$ be a cellular automaton. A k -substitution ξ is called k -invariant of A if $A^k \circ \xi = \xi \circ A$. The set of all k -invariants of the cellular automaton A is denoted by $\text{Inv}_k(A)$. The substitution ξ is called a regular k -invariant of A if ξ is a k -invariant and regular. The set of regular k -invariants is denoted by $\text{Inv}_k^r(A)$.

Remark If $k = 1$, then the 1-invariants of a cellular automaton A with local rule $\phi : \mathcal{S}^D \rightarrow \mathcal{S}$ are given by $\text{Inv}_1(A) = \{\zeta : \mathcal{S} \rightarrow \mathcal{S} \mid \phi(\zeta(r_1), \dots, \zeta(r_D)) = \zeta(\phi(r_1, \dots, r_D)), r_1, \dots, r_D \in \mathcal{S}\}$.

For linear k -Fermat cellular automata the coefficients of the substitution polynomial of a k -invariant ξ can be characterized via the local rule.

Theorem 4.5 If A is a linear k -Fermat cellular automaton, then $\text{Inv}_k(A) = \prod_{l=1}^k \text{Inv}_1(A)$.

Proof: Let $\underline{f} \in \mathcal{S}(X)$ and let $R(X) = \sum_{j=0}^D r_j X^j$ be the local rule of A . Since ξ is a k -invariant, we have that $R(X)^k \xi(\underline{f}) = \xi(R(X)\underline{f}(X))$. Since R is k -Fermat, the left side becomes $R(X^k) \xi(\underline{f}) = R(X^k) \sum_{j=0}^{(k-1)} X^j \underline{f}(X^k)$ (using the definition of ξ). Thus the coefficient of X^{kl+j} , $j \in [k]$, in $R(X^k) \xi(\underline{f})$ can be computed as $r_0 \xi_j(f_l) + r_1 \xi_j(f_{l-1}) + \dots + r_D \xi_j(f_{l-D})$. On the other hand, the same coefficient in $\xi(R(X)\underline{f}(X))$ turns out to be $\xi_j(r_0 f_l + \dots + r_D f_{l-D})$, i.e. $\xi_j \circ A = A \circ \xi_j$. Therefore, ξ is a k -invariant if and only if each ξ_j is a 1-invariant of A . •

Example 1. For $\mathcal{S} = \mathbb{Z}_2$ and $R(X) = 1 + X$, we have $\text{Inv}_1(A) = \{s \mapsto s, s \mapsto 0\}$ ($s \in \mathcal{S}$) and $\text{Inv}_1(A) = \text{Inv}_1^r(A)$. The automaton specified by this rule is 2^n -Fermat (for any $n \in \mathbb{N}$). Thus $\text{Inv}_2^r(A)$ contains 2^2 regular 2-invariant substitutions: $0 \mapsto 00, 1 \mapsto 00; 0 \mapsto 00, 1 \mapsto 10; 0 \mapsto 00, 1 \mapsto 01$ and $0 \mapsto 00, 1 \mapsto 11$. $\text{Inv}_4^r(A)$ contains 2^4 regular 4-invariants, among which $0 \mapsto 0000, 1 \mapsto 1011$.

2. For $\mathcal{S} = \mathbb{Z}_3$ and $R(X) = 2 + 2X$, we have $\text{Inv}_1(A) = \{s \mapsto \alpha s + \beta \mid \alpha, \beta \in \mathbb{Z}_3\}$ and $\text{Inv}_1^r(A) = \{s \mapsto \alpha s \mid \alpha \in \mathbb{Z}_3\}$. $\text{Inv}_3^r(A)$ contains 3^3 regular k -invariants, among which $0 \mapsto 000, 1 \mapsto 201, 2 \mapsto 102$.

3. $R(X) = 1 + 3X \in \mathbb{Z}_6[X]$ defines a 2-Fermat automaton such that $|\text{Inv}_1(A)| = 54$ and $|\text{Inv}_1^r(A)| = 18$. Thus there are 2^{18} regular 2-invariants.

In analogy with the substitution Ψ introduced before Theorem 3.2, we associate here with a linear k -Fermat automaton A , given by $R(X) = s + rX$, and a k -invariant $\xi \in \text{Inv}_k^r(A)$, a $(k \times k)$ -substitution $\Psi_\xi = \Psi(A, \xi) : \mathcal{S}^2(X, Y) \rightarrow \mathcal{S}^2(X, Y)$. Let $(a, b) \in \mathcal{S}^2$ and consider the polynomials $Q_m(X) = R(X)^m \xi(aX^{-1} + b) = \sum_{j \in \mathbb{Z}} q_{j,m} X^j$ for $m \in [k]$. For $m \in [k]$ the coefficients $q_{-1,m}, q_{0,m}, \dots, q_{k-1,m}$ of $Q_m(X)$ are uniquely determined by a and b , thus these coefficients can be regarded as functions from \mathcal{S}^2 to \mathcal{S} . Therefore, the substitution $\Psi_\xi = (\psi_{l,m})_{l,m \in [k]}$, defined as

$$\begin{aligned} \psi_{l,m} : \mathcal{S}^2 &\rightarrow \mathcal{S}^2 \\ (a, b) &\mapsto (q_{l-1,m}(a, b), q_{l,m}(a, b)) \end{aligned}$$

for $l, m \in [k]$, is well defined and called the *induced* substitution.

Remark 1. If ξ is a regular k -invariant, then the induced substitution Ψ_ξ is a regular substitution.

2. If the local rule of a linear cellular automaton is given by the polynomial $R(X) \in \mathcal{S}[X]$, of degree d , and if ξ is a regular k -invariant, then the induced substitution is defined by maps from \mathcal{S}^{d+1} which are constructed in a similar manner as for the case $d = 1$ considered above. $(d + 1)$ -tuples are then replaced by a $(k \times k)$ -array of $(d + 1)$ -tuples.

3. If A_ϕ is any cellular automaton such that

$$A_\phi(\sum f_j X^j) = \sum \phi(f_{j-d}, \dots, f_j) X^j$$

and with a k -invariant ξ , then the above construction yields an induced substitution as well.

Example For $\mathcal{S} = \mathbb{Z}_2$, the polynomial $R(X) = 1 + X$ defines a 2^2 -Fermat linear cellular automaton A . By Example 1 following Theorem 4.5, we have that the substitution ξ defined by the polynomial $P_\xi(X) = \text{id} + 0X + \text{id}X^2 + \text{id}X^3$ defines a 4-invariant of A . Consider a sequence $\dots zabcd\dots$, then substitution ξ transforms this into the top row shown below:

$$\begin{array}{cccc|cccc|cccc} \dots & z & 0 & z & z & a & 0 & a & a & \mathbf{b} & \mathbf{0} & \mathbf{b} & \mathbf{b} & c & 0 & c & c & \dots \\ \dots & & z & z & 0 & z & + & a & a & \mathbf{0} & \mathbf{a} & + & \mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{0} & b & + & c & c & c & 0 & \dots \\ \dots & & & 0 & z & z & + & a & z & 0 & \mathbf{a} & \mathbf{a} & + & \mathbf{b} & \mathbf{a} & \mathbf{0} & \mathbf{b} & b & + & c & b & 0 & c & \dots \\ \dots & & & & z & a & a & z & \mathbf{a} & \mathbf{b} & \mathbf{b} & \mathbf{a} & \mathbf{b} & c & c & b & c & \dots \end{array}$$

The other rows are part of the orbit generated with the given cellular automaton rule starting from the first row. The induced substitution Ψ_ξ can be read directly from these rows:

$$(a, b) \mapsto \begin{bmatrix} (a, b) & (b, 0) & (0, b) & (b, b) \\ (0, a + b) & (a + b, b) & (b, b) & (b, 0) \\ (a, a + b) & (a + b, a) & (a, 0) & (0, b) \\ (a, b) & (b, b) & (b, a) & (a, b) \end{bmatrix}. \quad (2)$$

The following theorem is a straightforward generalization of Theorem 3.2. It exploits the existence of k -invariants instead of the k -Fermat property.

Theorem 4.6 Let A_ϕ be a cellular automaton and let ξ be a regular k -invariant and let $\Psi_\xi = (\psi_{i,j})_{i,j \in [k]}$ be the induced k -substitution. There exists a natural number D , and embedding $\iota = \iota_D$, such that for $\underline{f} \in \mathcal{S}(X)$ with $\xi(\underline{f}) = \underline{f}$ and all $n \in \mathbb{N}$, $l \in [k]$ we have

$$A^{n k + l}(\underline{f}) = \mathbf{p}(\sigma_l(\iota(A^n(\underline{f})))) ,$$

where σ_l is the k -substitution $(\psi_{i,l})_{i \in [k]}$.

Proof. Since ξ is a k -invariant and $\underline{f} = \xi(\underline{f})$, we have $A^{kn}(\underline{f}) = A^{kn}(\xi(\underline{f})) = \xi(A^n(\underline{f}))$ for all $n \in \mathbb{N}$. That proves the assertion for $l = 0$. The assertion for $l \neq 0$ follows from the definition of the induced substitution. \bullet

We illustrate this theorem by continuing the above example, with $\mathcal{S} = \mathbb{Z}_2$,

$R(X) = 1 + X$ and the 4-invariant which produces the following substitutions:
 $0 \mapsto 0000$, $1 \mapsto 1011$. Take \underline{f} as the sequence which is the limit obtained from applying this substitution to $\cdots 0001011000 \cdots$ (the boldface element is at position 0). This yields the top row in

$$\begin{array}{l} \underline{f} \quad \cdots 00010110000101110110000000000000010111011 \cdots \\ A(\underline{f}) \cdots 000111010001110011010000000000000011100110 \cdots \end{array}$$

Observe that $\underline{f} = \xi(\underline{f})$. Theorem 4.6 states that, for example, row 6 ($= 1 \cdot 4 + 2$) in the orbit $O_A(\underline{f})$ can be obtained from $A(\underline{f})$ displayed above, by imbedding this last sequence in a sequence of pairs (corresponding to $D = 2$), as given by

$$\cdots (0, \underline{1})(\underline{1}, \underline{1})(1, \underline{1})(1, \underline{0})(0, \underline{1})(1, \underline{0})(0, \underline{0})(0, \underline{0})(0, \underline{1})(1, \underline{1})(1, \underline{1}) \cdots$$

(the last (underlined) components in these pairs are the elements of $A(\underline{f})$, the first components are the ones that immediately precede them). Then apply the substitutions displayed in row 2 of the induced substitution array Ψ_ξ given above (see(2): row 2 is actually the third row, as counting starts at zero). This gives

$$\cdots (0, \underline{1})(1, \underline{0})(0, \underline{0})(0, \underline{1})(1, \underline{0})(0, \underline{1})(1, \underline{0})(1, \underline{1})|(1, \underline{0})(0, \underline{1})(1, \underline{0})(1, \underline{1})|(1, \underline{1})(1, \underline{1}) \cdots$$

The last components in these pairs form row 6 in $O_A(\underline{f})$. In a similar way, using row l of the Ψ_ξ -substitution array, where $l \in \{0, 1, 2, 3\}$, starting from the embedding of row $A^n(\underline{f})$ in a sequence of pairs, would produce row $4 \cdot n + l$ in this orbit.

As a consequence of Theorem 2.4, we obtain

Corollary 4.7 *If ξ is a regular k -invariant of the cellular automaton A and if Ψ_ξ is the induced substitution, then the sequence $\left(\frac{1}{k^n} G(\Psi_\xi^n(\iota(\underline{f})))\right)_{n \in \mathbb{N}}$ is a Cauchy sequence for all $\underline{f} \in S_c(X)$.*

As a final step, we combine the notion of symmetry of a cellular automaton with the existence of induced substitutions. We are interested in the following problem. Suppose f_0, \dots, f_{N-1} is the top row of a ϕ -configuration of size N and assume that the configuration is, e.g., rotationally symmetric. If ξ is a regular k -invariant of A_ϕ , then $\xi(\sum f_j X^j)$ gives another top row of a ϕ -configuration of size kN ; what are the symmetry properties of the larger configuration?

Definition 4.8 *Let A_ϕ be a cellular automaton with local rule $\phi: S^2 \rightarrow S$ and let the substitution $\xi = (\xi_i)_{i \in [k]}$ be a k -invariant of A . The k -invariant is palindromic if $\xi_i = \xi_{k-i-1}$ holds for all $i \in [k]$.*

The k -invariant ξ is called rotationally symmetric if the induced substitution $\Psi_\xi = (\psi_{l,m})$ satisfies

$$\mathbf{p}(\psi_{i,0}(a, b)) = \mathbf{p}(\psi_{k-1,i}(a, b))$$

for all $(a, b) \in S^2$ and all $i \in [k]$.

If $\xi \in \text{Inv}_k(A)$ is palindromic and rotationally symmetric, then ξ is called totally symmetric.

An example of a rotationally symmetric 4-invariant is given by the example preceding Theorem 4.6: the last components of the rightmost column in the Ψ_ξ -array equals the last components of its top row.

Remark If $\xi = (\xi_l)$ is a k -invariant for A_ϕ and $\Psi_\xi = (\psi_{l,m})$ the induced substitution, then we have $\mathbf{p}(\psi_{i,0}(a,b)) = \xi_i(b)$ for $i \in [k]$ and there exist functions $\rho_i \in \text{Abb}(S)$, $i \in [k]$, such that $\mathbf{p}(\psi_{k-1,i}(a,b)) = \rho_i(b)$. In particular, if ξ is rotationally symmetric, then we have $\rho_i = \xi_i$ for all $i \in [k]$.

Lemma 4.9 ⁴ Let A be a cellular automaton with local rule $\phi : S^2 \rightarrow S$. If $f(X) \in S[X]$ is of degree $\leq N$, then $f(X)$ defines a rotationally symmetric ϕ -configuration of size $(N+1)$ if and only if for the coefficients $O_{i,j}$ of $O_A(f)(X,Y)$ the equations $O_{i,0} = O_{N,i}$ hold for $i = 0, \dots, N$.

In the linear case a carefully performed substitution can preserve symmetries.

Lemma 4.10 Let A_ϕ be a cellular automaton with rotationally symmetric local rule $\phi(x,y) = rx + sy$. If f_0, \dots, f_{N-1} is a top row of a rotationally symmetric ϕ -configuration of size N and if $\xi \in \text{Inv}_k^r(A_\phi)$ is rotationally symmetric, then g_0, \dots, g_{kN-1} , where $g(X) = \sum_{j \in \mathbb{Z}} g_j X^j = \xi(\sum_{i=0}^{N-1} f_i X^i)$, is the top row of a rotationally symmetric ϕ -configuration of size kN .

Proof. Let $f(X) = \sum_{j=0}^{N-1} f_j X^j$ and $\underline{Q} = O_A(f)(X,Y) = \sum O_{i,j} X^i Y^j$ its orbit. If Ψ_ξ denotes the induced $(k \times k)$ -substitution, then $\Psi_\xi(\underline{Q}) = \sum \Theta_{i,j} X^i Y^j$ is the orbit of $\xi(f)$. It remains to prove that $\Theta_{i,0} = \Theta_{kN-1,i}$ holds for all $i = 0, \dots, kN-1$.

By Lemma 4.9, we have $O_{i,0} = O_{N-1,i}$ for $i = 0, \dots, N-1$. Since ξ is rotationally symmetric, we have $\Theta_{kj+l,0} = \xi_l(O_{j,0})$ for $kj+l \in [Nk]$ and $l \in [k]$ and $\Theta_{kN-1,kj+l} = \xi_l(O_{N-1,j}) = \xi_l(O_{j,0})$, which proves the assertion. •

Lemma 4.10 remains true if rotationally symmetric is replaced by totally symmetric.

We can now establish the existence of fractals with prescribed symmetries.

Theorem 4.11 Let A_ϕ , with $\phi(x,y) = rx + sy$, define a cellular automaton with a rotationally (totally) symmetric linear local rule. Let $f(X) \in S[X]$ of degree $\leq N$ define a rotationally (totally) symmetric ϕ -configuration of size $(N+1)$ and let $\xi \in \text{Inv}_k^r(A_\phi)$ be a regular rotationally (totally) symmetric k -invariant. If $\iota(O_A(f)) = \iota_2(O_A(f))$ denotes the embedding of the orbit $O_A(f)$ which replaces $(O_A(f))_{j,i}$ by $((O_A(f))_{j-1,i}, (O_A(f))_{j,i})$, and if Ψ_ξ denotes the induced substitution, then the sequence

$$\left(\frac{1}{k^n(N+1)} \begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix} \mathbf{G}(\Psi_\xi^n(\iota(O_A(f)))) \cap \nabla \right)_{n \in \mathbb{N}},$$

where ∇ is the equilateral triangle given by the points $(0,0)$, $(1,0)$ and $(1/2, \sqrt{3}/2)$ (ordinate axis positively oriented downwards), is a Cauchy sequence. The limit, denoted as $\mathcal{X} = \mathcal{X}(f, \xi)$, is a compact set in ∇ and \mathcal{X} is rotationally (totally) symmetric.

Proof. By Theorem 2.4, the limit $1/k^n \mathbf{G}(\Psi_\xi^n(\iota(f)))$ exists and is contained in $[0, N+1]^2$. The factor transforms the triangle $\{(x,y) \in [0, N+1]^2 \mid y \leq x\}$ into an equilateral triangle of size 1. Since the limit is independent of the particular graphical representation ("pixel-shape"), we can define $\gamma : S^2 \setminus \{0\} \rightarrow \mathcal{H}(\mathbb{R}^2)$ by setting $\gamma(s_{-1}, s_0) = \cup \{(j,0) \mid s_j \neq 0; j = -1, 0\}$. Then the rescaled version of $G_\gamma(\Psi_\xi^n(\iota(f))) \cap \nabla$ is rotationally (totally) symmetric for each $n \in \mathbb{N}$. Thus the limit is rotationally (totally) symmetric. •

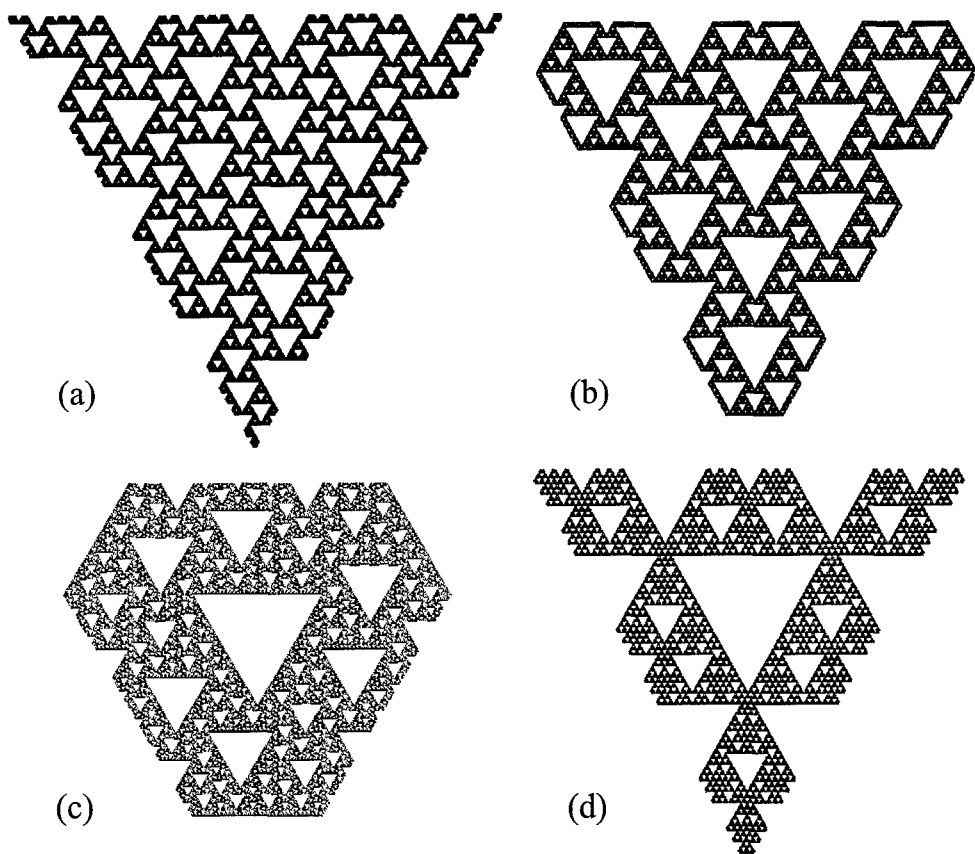


Figure 2. Symmetric fractals generated by cellular automata. (a) and (b): states in $\mathbb{Z}_2 = \{0,1\}$ with local rule $\phi(x,y) = x + y$ and, for (a): with top row generated from the initial top-row configuration 1101101101 and the 16-substitution $0 \mapsto 0000000000000000$, $1 \mapsto 0110011000000110$; for (b): initial configuration 0110110110 and 8-substitution $0 \mapsto 00000000$, $1 \mapsto 01111110$. (c): states in $\mathbb{F}_4 = \{0,1,\zeta,1+\zeta\} = \{0,1,2,3\}$ with local rule $\phi(x,y) = \zeta x + (1+\zeta)y$, initial state 0130 and 4-substitution $0 \mapsto 0000$, $1 \mapsto 1021$, $2 \mapsto 2032$, $3 \mapsto 3013$. (d): states in $\mathbb{Z}_3 = \{0,1,2\}$ with local rule $\phi(x,y) = 2x + 2y$, initial configuration 11011011 and 9-substitution $0 \mapsto 000000000$, $1 \mapsto 002020200$, $2 \mapsto 001010100$. (a) and (c) are rotationally symmetric, (b) and (d) are totally symmetric. The Hausdorff dimensions are respectively $\log 117 / \log 16 \approx 1.717$, $\log 36 / \log 8 \approx 1.723$, $\log 13 / \log 4 \approx 1.850$, $\log 54 / \log 9 \approx 1.815$.

Figure 2 shows a few examples of symmetric fractal limits obtained from rescaled cellular automata orbits generated by a proper initial top-row configuration subjected to a proper substitution as presented in the above theorem.

5 Conclusions

The above described method allows one to construct fractals which reflect the local rule of a cellular automaton as well as showing a global symmetry. Since these fractals are generated by a $(k \times k)$ -substitution, these fractals can also be described in terms of hierarchical iterated function systems, in particular by so called k -adic HIFS⁸. By introducing a transition matrix⁸ for the generating substitution it is possible to compute the fractal's box-counting dimension^{5,6} which coincides with the Hausdorff dimension.

Acknowledgment

This research was supported by the Research Council of the Katholieke Universiteit Leuven, by the Concerted Action Project GOA-MIPS of the Flemish Community, and by the Belgian Program on Interuniversity Attraction Poles of the Belgian Prime Minister's Office for Science, Technology and Culture (IUAP P4-02).

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